Lecture 17: Subspaces, basis, dimensions, row-space algorithm

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15.2 Subspace test

Lemma Null(A) is a subspace of $\mathbb{R}^n$.

PROOF

We run the subspace test:

1) contains zero/$0 \in$ Null(A)?
   $\rightarrow A \cdot 0 = 0$, so $0 \in$ Null(A)
2) closed under addition?
   $\rightarrow$ Let x, y, Null(A).
   Then $A(x+y) = Ax + Ay = 0 + 0 = 0, \Rightarrow x + y \in$ Null(A)
3) closed under scalar multiplication?
   $\rightarrow$ Let $x \in$ Null(A) and $k \in \mathbb{R}$. Then, $A(kx) = k(Ax) = k \cdot 0 = 0 \Rightarrow kx \in$ Null(A)

15.3 Basis and dimension

$A = \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$, find a basis of Null(A)

$$= \left[\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & -2 & 2 & -4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$Null(A) = S = \left\{ \begin{pmatrix} -s-t \\ -s+2t \\ s \\ t \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$$

$$= \left\{ s \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}$$

$$= span \underbrace{\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}}_{\text{this is already a basis!}}$$

(look at 3rd and 4th entry)

Here, dim(Null(A))=2, and rank(A)=2.

Since dim(Null(A))= # of columns without leading ones and $rank(A) =$ # columns with leading ones, we obtain:

Rank-Nullity Theorem

$\dim(Null(A)) + rank(A) =$ # of columns of A

15.4 Inhomogeneous linear systems

Use A from 15.3 and let $b = \begin{bmatrix} 10 \\ 3 \\ 4 \end{bmatrix}$. Solve $Ax = b$,

that is: $= \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 4 \end{bmatrix}$

$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 10 \end{array}\right.$ $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 10 \end{array}\right.$

$$\begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \quad \begin{bmatrix} 4 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 10 \\ 0 & 1 & 1 & -2 & 3 \\ 1 & 0 & 1 & 1 & 4 \end{array}\right] \sim \dots \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 10 \\ 0 & 1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right]$$

$$S = \left\{ \left. \begin{pmatrix} 4-s-t \\ 3-s+2t \\ s \\ t \end{pmatrix} \right| s,t \in \mathbb{R} \right\}$$

$$= \left\{ \left. \underbrace{\begin{pmatrix} 4 \\ 3 \\ 0 \\ 0 \end{pmatrix}}_{\text{particular solution}} + \underbrace{s \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}}_{\text{all solutions to } Ax=0 \ (= Null(A))} \right| s,t \in \mathbb{R} \right\}$$

Theorem

If $Ax = b$ is consisten and v is a particular solution that is $Av = b$, then the general solution is:

$$S = v + Null(A)$$

15.5 Summary of facts

a) consistency of linear systems

Let $A \in \mathrm{M}_{mn}(\mathbb{R})$ and $b \in \mathbb{R}^m$.
Linear system $Ax = b$ is consistent
<=> $b \in$ Col(A)
<=>$rank(A) = rank(A|b)$

Linear system $Ax = b$ is consistent for every $b \in \mathbb{R}^m$
<=>every $b \in \mathbb{R}^m$ is in Col(A)
<=>Col(A)=$\mathbb{R}^m$
<=>rank(A)=m

b) number of solutions

Let $A \in \mathrm{M}_{mn}(\mathbb{R})$ and $b \in \mathbb{R}^m$ so that $Ax = b$ is consistent.
Linear system $Ax = b$ has a unique solution:
<=>no free parameters in the general solution
<=>every column of the RREF of A has a leading one
<=>linear system $Ax = 0$ has the unique solution $x - 0$
<=>columns of A are linearly independent
<=>Null(A)= {0}
<=>dim(Null(A))= 0
<=>rank(A)=n *see rank-nullity theorem

Message

We can solve $Ax = b$ for every b **iff** rank(A)=m. Such a solution is unique **iff** rank(A)=n.

16 Finding bases

Let $W = span\{v_1, v_2, \dots, v_k\}$. Find a basis of W.

16.1 Row space algorithm

1) Write vectors as rows into matrix
2) Transform into any REF (or into RREF if you like)
3) Non-zero rows form a basis of W

Example

$$W = span\left\{\begin{pmatrix}1\\2\\2\\3\end{pmatrix},\begin{pmatrix}2\\-2\\-8\\4\end{pmatrix},\begin{pmatrix}1\\1\\0\\1\end{pmatrix},\begin{pmatrix}0\\2\\4\\1\end{pmatrix}\right\}$$

$$(1)\begin{bmatrix}1&2&2&3\\2&-2&-8&4\\1&1&0&1\\0&2&4&1\end{bmatrix}\ (2)\sim\underbrace{\begin{bmatrix}1&0&-2&0\\0&1&2&0\\0&0&0&1\\0&0&0&0\end{bmatrix}}_{\text{(R)REF}}\ (3)\ \underbrace{\left\{\begin{pmatrix}1\\0\\-2\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}}_{\text{basis of W}}$$

16.2 Extending linearly independent sets to a basis of $\mathbb{R}^n$

1) Write vectors as rows into matrix.
2) Transform into any REF (or RREF).
3) If the k-th column has no leading one, add the vector $\begin{pmatrix}0\\\vdots\\1\\\vdots\\0\end{pmatrix}$ to the LI set.

1 in k th entry

This yeilds a basis of $\mathbb{R}^n$.

Example from above

(1) and (2) as above, (3) Basis of $\mathbb{R}^n$

$$=\left\{\begin{pmatrix}1\\0\\-2\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}$$

16.3 Reducing spanning set to a basis of W

There is a problem in 16.1. The basis doesn't involve any of the initial vectors. What if we wanted to keep them and just wanted to reduce the spanning set until we obtain a basis? We use the **column space algorithm**.

1) write vectors in columns of matrix
2) Transform into any REF (or RREF).
3) Keep the vectors that correspond to columns with leading ones.

Example

$$(1)\begin{bmatrix}1&2&1&0\\2&-2&1&2\\2&-8&0&4\\3&4&1&1\end{bmatrix}\quad (2)\begin{array}{c}\begin{array}{cccc}\text{1st}&\text{2nd}&\text{3rd}&\end{array}\\ \begin{bmatrix}1&2&1&0\\0&1&1/6&\frac{-1}{3}\\0&0&1&\frac{-1}{5}\\0&0&0&0\end{bmatrix}\end{array}\quad (3)\ \underbrace{\left\{\begin{pmatrix}1\\2\\2\\3\end{pmatrix},\begin{pmatrix}2\\-2\\-8\\4\end{pmatrix},\begin{pmatrix}1\\1\\0\\1\end{pmatrix}\right\}}_{\text{Basis of W}}$$

And so:

$$\dim\big(Row(A)\big) = rank(A)$$
$$\dim\big(Col(A)\big) = rank(A)$$

$$\dim\big(Row(A)\big) = \dim(Col(A))$$